Exercise 2.5.6

Solve Laplace's equation inside a semicircle of radius a ($0 < r < a$, $0 < \theta < \pi$) subject to the boundary conditions [*Hint*: In polar coordinates,

$$\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0,$$

it is known that if $u(r,\theta) = \phi(\theta)G(r)$, then $\frac{r}{G}\frac{d}{dr}\left(r\frac{dG}{dr}\right) = -\frac{1}{\phi}\frac{d^2\phi}{d\theta^2}$.]:

(a) $u = 0$ on the diameter and $u(a,\theta) = g(\theta)$

(b) the diameter is insulated and $u(a,\theta) = g(\theta)$

Solution

Because the Laplace equation and all but one of its associated boundary conditions are linear and homogeneous, the method of separation of variables can be applied to solve it. Assume a product solution of the form $u(r,\theta) = R(r)\Theta(\theta)$ and plug it into the PDE.

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial}{\partial r}R(r)\Theta(\theta)\right] + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}R(r)\Theta(\theta) = 0$$

$$\frac{\Theta(\theta)}{r}\frac{d}{dr}\left(r\frac{dR}{dr}\right) + \frac{R(r)}{r^2}\frac{d^2\Theta}{d\theta^2} = 0$$

Multiply both sides by $r^2/[R(r)\Theta(\theta)]$ in order to separate variables.

$$\frac{r}{R(r)}\frac{d}{dr}\left(r\frac{dR}{dr}\right) + \frac{1}{\Theta(\theta)}\frac{d^2\Theta}{d\theta^2} = 0$$

$$\frac{r}{R(r)}\frac{d}{dr}\left(r\frac{dR}{dr}\right) = -\frac{1}{\Theta(\theta)}\frac{d^2\Theta}{d\theta^2}$$

The only way a function of r can be equal to a function of θ is if both are equal to a constant λ.

$$\frac{r}{R(r)}\frac{d}{dr}\left(r\frac{dR}{dr}\right) = -\frac{1}{\Theta(\theta)}\frac{d^2\Theta}{d\theta^2} = \lambda$$

As a result of separating variables, the PDE has reduced to two ODEs—one in each independent variable.

$$\left.\begin{aligned} \frac{r}{R}\frac{d}{dr}\left(r\frac{dR}{dr}\right) &= \lambda \\ -\frac{1}{\Theta}\frac{d^2\Theta}{d\theta^2} &= \lambda \end{aligned}\right\}$$

Values of λ for which nontrivial solutions to these ODEs and the associated boundary conditions exist are called eigenvalues, and the solutions themselves are called eigenfunctions. Note that it doesn't matter whether the minus sign is grouped with r or θ as long as all eigenvalues are taken into account.

Part (a)

Substitute the product solution $u(r,\theta) = R(r)\Theta(\theta)$ into the homogeneous boundary conditions.

$$\begin{aligned} u(r,0) = 0 \quad &\rightarrow \quad R(r)\Theta(0) = 0 \quad &\rightarrow \quad \Theta(0) = 0 \\ u(r,\pi) = 0 \quad &\rightarrow \quad R(r)\Theta(\pi) = 0 \quad &\rightarrow \quad \Theta(\pi) = 0 \end{aligned}$$

Solve the ODE for Θ.

$$\frac{d^2\Theta}{d\theta^2} = -\lambda\Theta$$

Check to see whether there are positive eigenvalues: $\lambda = \mu^2$.

$$\frac{d^2\Theta}{d\theta^2} = -\mu^2\Theta$$

The general solution can be written in terms of sine and cosine.

$$\Theta(\theta) = C_1 \cos\mu\theta + C_2 \sin\mu\theta$$

Apply the boundary conditions to determine C_1 and C_2.

$$\begin{aligned} \Theta(0) &= C_1 = 0 \\ \Theta(\pi) &= C_1 \cos\mu\pi + C_2 \sin\mu\pi = 0 \end{aligned}$$

This first equation makes the second equation reduce to $C_2 \sin\mu\pi = 0$. To avoid the trivial solution, we insist that $C_2 \neq 0$.

$$\begin{aligned} \sin\mu\pi &= 0 \\ \mu\pi &= n\pi, \quad n = 1, 2, \ldots \\ \mu &= n \end{aligned}$$

There are positive eigenvalues $\lambda = n^2$, and the eigenfunctions associated with them are

$$\Theta(\theta) = C_2 \sin\mu\theta \quad \rightarrow \quad \Theta_n(\theta) = \sin n\theta.$$

Note that $n = 0$ is not considered because that would lead to the zero eigenvalue, and negative integer values of n aren't considered because they lead to redundant eigenvalues. Using $\lambda = n^2$, solve the ODE for R now.

$$\frac{r}{R}\frac{d}{dr}\left(r\frac{dR}{dr}\right) = n^2$$

Expand the left side.

$$\frac{r}{R}(R' + rR'') = n^2$$

Multiply both sides by R and bring all terms to the left side.

$$r^2R'' + rR' - n^2R = 0$$

This is an equidimensional ODE, so it has solutions of the form $R(r) = r^m$.

$$R = r^m \quad \rightarrow \quad R' = mr^{m-1} \quad \rightarrow \quad R'' = m(m-1)r^{m-2}$$

Substitute these formulas into the ODE and solve the resulting equation for m.

$$r^2 m(m-1)r^{m-2} + rmr^{m-1} - n^2 r^m = 0$$

$$m(m-1)r^m + mr^m - n^2 r^m = 0$$

$$m(m-1) + m - n^2 = 0$$

$$m^2 - n^2 = 0$$

$$(m+n)(m-n) = 0$$

$$m = \{-n, n\}$$

Two solutions to the ODE are $R = r^{-n}$ and $R = r^n$. By the principle of superposition, the general solution for R is a linear combination of these two.

$$R(r) = Ar^{-n} + Br^n$$

Now check to see if zero is an eigenvalue: $\lambda = 0$.

$$\frac{d^2\Theta}{d\theta^2} = 0$$

The general solution is a straight line.

$$\Theta(\theta) = C_3\theta + C_4$$

Apply the two boundary conditions to determine C_3 and C_4.

$$\Theta(0) = C_4 = 0$$
$$\Theta(\pi) = C_3\pi + C_4 = 0$$

This first equation makes the second equation reduce to $C_3\pi = 0$, which means $C_3 = 0$.

$$\Theta(\theta) = 0$$

The trivial solution is obtained, so zero is not an eigenvalue. Check to see if there are negative eigenvalues: $\lambda = -\gamma^2$.

$$\frac{d^2\Theta}{d\theta^2} = \gamma^2\Theta$$

The general solution can be written in terms of hyperbolic sine and hyperbolic cosine.

$$\Theta(\theta) = C_5 \cosh\gamma\theta + C_6 \sinh\gamma\theta$$

Apply the two boundary conditions to determine C_5 and C_6.

$$\Theta(0) = C_5 = 0$$
$$\Theta(\pi) = C_5 \cosh\gamma\pi + C_6 \sinh\gamma\pi = 0$$

This first equation makes the second equation reduce to $C_6 \sinh\gamma\pi = 0$. No nonzero value of γ can satisfy this equation, so $C_6 = 0$.

$$\Theta(\theta) = 0$$

The trivial solution is obtained, so there are no negative eigenvalues. According to the principle of superposition, the general solution to the PDE is a linear combination of the eigenfunctions over all the eigenvalues.

$$u(r,\theta) = \sum_{n=1}^{\infty}(A_n r^{-n} + B_n r^n)\sin n\theta$$

For the solution to remain finite as $r \to 0$, set $A_n = 0$.

$$\boxed{u(r,\theta) = \sum_{n=1}^{\infty} B_n r^n \sin n\theta}$$

Use the boundary condition at $r = a$ to determine the remaining constants B_n.

$$u(a,\theta) = \sum_{n=1}^{\infty} B_n a^n \sin n\theta = g(\theta)$$

To find B_n, multiply both sides by $\sin p\theta$, where p is an integer.

$$\sum_{n=1}^{\infty} B_n a^n \sin n\theta \sin p\theta = g(\theta)\sin p\theta$$

Integrate both sides with respect to θ from 0 to π.

$$\int_0^{\pi} \sum_{n=1}^{\infty} B_n a^n \sin n\theta \sin p\theta \, d\theta = \int_0^{\pi} g(\theta)\sin p\theta \, d\theta$$

Bring the constants in front.

$$\sum_{n=1}^{\infty} B_n a^n \int_0^{\pi} \sin n\theta \sin p\theta \, d\theta = \int_0^{\pi} g(\theta)\sin p\theta \, d\theta$$

Because the sine functions are orthogonal, this integral on the left is zero if $n \neq p$. Only if $n = p$ does the integral yield a nonzero result.

$$B_n a^n \int_0^{\pi} \sin^2 n\theta \, d\theta = \int_0^{\pi} g(\theta)\sin n\theta \, d\theta$$

$$B_n a^n \left(\frac{\pi}{2}\right) = \int_0^{\pi} g(\theta)\sin n\theta \, d\theta$$

Therefore,

$$\boxed{B_n = \frac{2}{\pi a^n}\int_0^{\pi} g(\theta)\sin n\theta \, d\theta.}$$

Part (b)

An insulated diameter implies that there are homogeneous Neumann boundary conditions along $\theta = 0$ and $\theta = \pi$. Substitute the product solution $u(r,\theta) = R(r)\Theta(\theta)$ into them.

$$\begin{aligned} \frac{\partial u}{\partial \theta}(r,0) = 0 \quad &\rightarrow \quad R(r)\Theta'(0) = 0 \quad \rightarrow \quad \Theta'(0) = 0 \\ \frac{\partial u}{\partial \theta}(r,\pi) = 0 \quad &\rightarrow \quad R(r)\Theta'(\pi) = 0 \quad \rightarrow \quad \Theta'(\pi) = 0 \end{aligned}$$

Solve the ODE for Θ.

$$\frac{d^2\Theta}{d\theta^2} = -\lambda\Theta$$

Check to see whether there are positive eigenvalues: $\lambda = \mu^2$.

$$\frac{d^2\Theta}{d\theta^2} = -\mu^2\Theta$$

The general solution can be written in terms of sine and cosine.

$$\Theta(\theta) = C_1 \cos\mu\theta + C_2 \sin\mu\theta$$

Differentiate it with respect to θ.

$$\Theta'(\theta) = \mu(-C_1 \sin\mu\theta + C_2 \cos\mu\theta)$$

Apply the boundary conditions to determine C_1 and C_2.

$$\begin{aligned} \Theta'(0) &= \mu(C_2) = 0 \\ \Theta'(\pi) &= \mu(-C_1 \sin\mu\pi + C_2 \cos\mu\pi) = 0 \end{aligned}$$

This first equation gives $C_2 = 0$, which makes the second equation reduce to $-C_1\mu\sin\mu\pi = 0$. To avoid the trivial solution, we insist that $C_1 \neq 0$.

$$\begin{aligned} \sin\mu\pi &= 0 \\ \mu\pi &= n\pi, \quad n = 1, 2, \ldots \\ \mu &= n \end{aligned}$$

There are positive eigenvalues $\lambda = n^2$, and the eigenfunctions associated with them are

$$\Theta(\theta) = C_1 \cos\mu\theta \quad \rightarrow \quad \Theta_n(\theta) = \cos n\theta.$$

Using $\lambda = n^2$, solve the ODE for R now.

$$\frac{r}{R}\frac{d}{dr}\left(r\frac{dR}{dr}\right) = n^2$$

Expand the left side.

$$\frac{r}{R}(R' + rR'') = n^2$$

Multiply both sides by R and bring all terms to the left side.

$$r^2R'' + rR' - n^2R = 0$$

This is an equidimensional ODE, so it has solutions of the form $R(r) = r^m$.

$$R = r^m \quad \rightarrow \quad R' = mr^{m-1} \quad \rightarrow \quad R'' = m(m-1)r^{m-2}$$

Substitute these formulas into the ODE and solve the resulting equation for m.

$$r^2 m(m-1)r^{m-2} + rmr^{m-1} - n^2 r^m = 0$$

$$m(m-1)r^m + mr^m - n^2 r^m = 0$$

$$m(m-1) + m - n^2 = 0$$

$$m^2 - n^2 = 0$$

$$(m+n)(m-n) = 0$$

$$m = \{-n, n\}$$

Two solutions to the ODE are $R = r^{-n}$ and $R = r^n$. By the principle of superposition, the general solution for R is a linear combination of these two.

$$R(r) = Ar^{-n} + Br^n$$

Now check to see if zero is an eigenvalue: $\lambda = 0$.

$$\frac{d^2\Theta}{d\theta^2} = 0$$

The general solution is a straight line.

$$\Theta(\theta) = C_3\theta + C_4$$

Apply the two boundary conditions to determine C_3 and C_4.

$$\Theta'(0) = C_3 = 0$$
$$\Theta'(\pi) = C_3 = 0$$

C_4 remains arbitrary.

$$\Theta(\theta) = C_4$$

The trivial solution is not obtained, so zero is an eigenvalue. Using $\lambda = 0$, solve the ODE for R now.

$$\frac{r}{R}\frac{d}{dr}\left(r\frac{dR}{dr}\right) = 0$$

Multiply both sides by R/r.

$$\frac{d}{dr}\left(r\frac{dR}{dr}\right) = 0$$

Integrate both sides with respect to r.

$$r\frac{dR}{dr} = D_1$$

Divide both sides by r.

$$\frac{dR}{dr} = \frac{D_1}{r}$$

Integrate both sides with respect to r once more.

$$R(r) = D_1 \ln r + D_2$$

Check to see if there are negative eigenvalues: $\lambda = -\gamma^2$.

$$\frac{d^2\Theta}{d\theta^2} = \gamma^2\Theta$$

The general solution can be written in terms of hyperbolic sine and hyperbolic cosine.

$$\Theta(\theta) = C_5 \cosh \gamma\theta + C_6 \sinh \gamma\theta$$

Differentiate it with respect to θ.

$$\Theta'(\theta) = \gamma(C_5 \sinh \gamma\theta + C_6 \cosh \gamma\theta)$$

Apply the two boundary conditions to determine C_5 and C_6.

$$\begin{aligned} \Theta'(0) &= \gamma(C_6) = 0 \\ \Theta'(\pi) &= \gamma(C_5 \sinh \gamma\pi + C_6 \cosh \gamma\pi) = 0 \end{aligned}$$

This first equation gives $C_6 = 0$, which makes the second equation reduce to $C_5\gamma \sinh \gamma\pi = 0$. No nonzero value of γ can satisfy this equation, so $C_5 = 0$.

$$\Theta(\theta) = 0$$

The trivial solution is obtained, so there are no negative eigenvalues. According to the principle of superposition, the general solution to the PDE is a linear combination of the eigenfunctions over all the eigenvalues.

$$u(r,\theta) = (A_0 \ln r + B_0) + \sum_{n=1}^{\infty} (A_n r^{-n} + B_n r^n) \cos n\theta$$

For the solution to remain finite as $r \to 0$, set $A_0 = 0$ and $A_n = 0$.

$$\boxed{u(r,\theta) = B_0 + \sum_{n=1}^{\infty} B_n r^n \cos n\theta}$$

Use the boundary condition at $r = a$ to determine the remaining constants, B_0 and B_n.

$$u(a,\theta) = B_0 + \sum_{n=1}^{\infty} B_n a^n \cos n\theta = g(\theta) \tag{1}$$

To find B_0, integrate both sides with respect to θ from 0 to π.

$$\int_0^{\pi} \left(B_0 + \sum_{n=1}^{\infty} B_n a^n \cos n\theta \right) d\theta = \int_0^{\pi} g(\theta)\, d\theta$$

Split up the integral on the left side and bring the constants in front.

$$B_0 \underbrace{\int_0^{\pi} d\theta}_{=\,\pi} + \sum_{n=1}^{\infty} B_n a^n \underbrace{\int_0^{\pi} \cos n\theta \, d\theta}_{=\,0} = \int_0^{\pi} g(\theta)\, d\theta$$

$$B_0(\pi) = \int_0^\pi g(\theta)\,d\theta$$

Therefore,

$$\boxed{B_0 = \frac{1}{\pi}\int_0^\pi g(\theta)\,d\theta.}$$

To get B_n, multiply both sides of equation (1) by $\cos p\theta$, where p is an integer.

$$B_0 \cos p\theta + \sum_{n=1}^\infty B_n a^n \cos n\theta \cos p\theta = g(\theta)\cos p\theta$$

Integrate both sides with respect to θ from 0 to π.

$$\int_0^\pi \left(B_0 \cos p\theta + \sum_{n=1}^\infty B_n a^n \cos n\theta \cos p\theta\right) d\theta = \int_0^\pi g(\theta)\cos p\theta\,d\theta$$

Split up the integral and bring the constants in front.

$$B_0 \underbrace{\int_0^\pi \cos p\theta\,d\theta}_{=\,0} + \sum_{n=1}^\infty B_n a^n \int_0^\pi \cos n\theta \cos p\theta\,d\theta = \int_0^\pi g(\theta)\cos p\theta\,d\theta$$

Because the cosine functions are orthogonal, this second integral on the left is zero if $n \neq p$. Only if $n = p$ does the integral yield a nonzero result.

$$B_n a^n \int_0^\pi \cos^2 n\theta\,d\theta = \int_0^\pi g(\theta)\cos n\theta\,d\theta$$

$$B_n a^n \left(\frac{\pi}{2}\right) = \int_0^\pi g(\theta)\cos n\theta\,d\theta$$

Therefore,

$$\boxed{B_n = \frac{2}{\pi a^n}\int_0^\pi g(\theta)\cos n\theta\,d\theta.}$$